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No fluctuation approximation in any desired precision for univariate function matrix representations

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Abstract The operator involving problems are mostly handled by using the matrix representations of the operators over a finite set of appropriately chosen basis functions in a Hilbert space as long as the related problem permits. The algebraic operator which multiplies its operand by a function is the focus of this work. We deal with the univariate case for simplicity. We show that a rapidly converging scheme can be constructed by defining an appropriate fluctuation operator which projects, in fact, to the complement of the space spanned by appropriately chosen finite number of basis functions. What we obtain here can be used in efficient numerical integration also.

Keywords Matrix representation · Fluctuation expansion · Hilbert spaces · Projection operators · Algebraic multiplication operators

1 Introduction

Matrix representations of linear operators have great importance in the solutions of many linear problems like ordinary differential equations, partial differential equations with appropriate initial and/or boundary conditions. These generally arise as inversion or eigenpair determination problems. The area of mathematical chemistry covers many applications amongst these problems. The algebraic operators which multiply their operands by certain given functions' values at a specific independent variable value are at the focus here. Although the function may be univariate or multivariate we prefer to deal with only univariate ones for simplicity. This does not bring any remarkable limitation since the derivation of the formulae here can be easily extended to the multivariate case. We are going to consider the linear operator $\widehat{f}$ as a mapping,

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from a denumerably infinite dimensional Hilbert space $\mathscr{H}$ over the functions which are continuous and therefore square integrable on a given closed interval $[a, b]$ and under a given weight function, to the same space. The explicit definition of this operator is as follows

$$\widehat{f}g(x) \equiv f(x)g(x), \quad x \in [a, b] \tag{1}$$

where $g(x)$ stands for any given function in $\mathscr{H}$. To get the chance of dealing with a universal entity which does not depend on the structure of the function $f(x)$ we define the following operator

$$\widehat{x}g(x) \equiv xg(x), \quad x \in [a, b] \tag{2}$$

where $g(x)$ represents any given function from $\mathscr{H}$ as before. This new definition enables us to write

$$\widehat{f} = f(\widehat{x}) \tag{3}$$

Now we can consider the orthonormal basis function set for $\mathscr{H}$ as follows

$$\mathscr{U} \equiv \{u_i(x)\}_{i=1}^{\infty} \tag{4}$$

which can be constructed from the ascending natural number powers of the independent variable via an appropriate orthonormalization procedure like Gram–Schmidt or Cholesky method. The operator $\widehat{f}$ has an infinite matrix representation on $\mathscr{H}$, whose general term at the intersection of the ith row and jth column is given below

$$\left[\mathbf{M}(\widehat{f})\right]_{ij} \equiv \left(u_i, \widehat{f}u_j\right), \quad i, j = 1, 2, \ldots \tag{5}$$

where $\mathbf{M}$ stands for the mapping (the explicit definition and certain properties of its restriction over a finite subspace will be given a little bit later) from the space of linear operators between $\mathscr{H}$ and $\mathscr{H}$ to the matrices denoting the matrix representation of the operator under consideration, and, we have used $\mathscr{H}$'s inner product whose explicit definition is as follows

$$(g_1, g_2) \equiv \int_a^b dx w(x) g_1(x) g_2(x), \quad g_1(x), g_2(x) \in \mathscr{H} \tag{6}$$

and $w(x)$ stands for a given weight function in the last equality.

Since the general tendency to approximate this infinite matrix representation is to truncate it as a block from its left uppermost part and then to tune the truncation order to get a desired approximation unless a very specific structure to get analytic results in the matrix representation exists, we define the n-dimensional subspace, $\mathscr{H}_n$, of $\mathscr{H}$ over the following basis set

$$\mathscr{U}_n \equiv \{u_i(x)\}_{i=1}^{n}, \quad n = 1, 2, \ldots \tag{7}$$

We can uniquely decompose the operator $\widehat{f}$ to four components characterizing transitions amongst $\mathscr{H}_n$ and its complement to the whole space $\mathscr{H}$, that is, $(\mathscr{H}-\mathscr{H}_n)$ as follows

$$\widehat{f} \equiv \widehat{f}_{ss} + \widehat{f}_{sc} + \widehat{f}_{cs} + \widehat{f}_{cc} \tag{8}$$

where the subindices s and c stand for recalling subspace $\mathscr{H}_n$ and its complement $(\mathscr{H}-\mathscr{H}_n)$ respectively. The right hand side components are defined as follows

$$\begin{array}{ll} \widehat{f}_{ss} \equiv \widehat{P}^{(n)}\widehat{f}\widehat{P}^{(n)}, & \widehat{f}_{sc} \equiv \widehat{P}^{(n)}\widehat{f}\left[\widehat{I}-\widehat{P}^{(n)}\right], \\ \widehat{f}_{cs} \equiv \left[\widehat{I}-\widehat{P}^{(n)}\right]\widehat{f}\widehat{P}^{(n)}, & \widehat{f}_{cc} \equiv \left[\widehat{I}-\widehat{P}^{(n)}\right]\widehat{f}\left[\widehat{I}-\widehat{P}^{(n)}\right] \end{array} \tag{9}$$

where $\widehat{I}$ denotes the unit operator of $\mathscr{H}$and $\widehat{P}^{(n)}$ is explicitly defined below

$$\widehat{P}^{(n)}g(x) \equiv \sum_{i=1}^{n}(u_i, g)\,u_i(x), \qquad n = 1, 2, \ldots \quad g(x) \in \mathscr{H} \tag{10}$$

As can be noticed immediately, $\widehat{P}^{(n)}$ is the unit operator for $\widehat{H}_n$ and it projects from $\mathscr{H}$ to $\mathscr{H}_n$. Its complementary companion $[\widehat{I}-\widehat{P}^{(n)}]$ is the unit operator for $(\mathscr{H}-\mathscr{H}_n)$ and projects from $\mathscr{H}$ to $(\mathscr{H}-\mathscr{H}_n)$. In the light of these informations, any decomposition component of $\widehat{f}$ at the right hand side of (8) maps from the subspace characterized by the second subindex to the subspace characterized by the first subindex of the component. Amongst these components, $\widehat{f}_{ss}$ is the most important one which maps from $\widehat{H}_n$ to $\widehat{H}_n$. Its matrix representation is just what we want to get as an approximation to the matrix representation of $\widehat{f}$.

Now we are ready to deal with the matrix representation related issues more intensely. As a first step we consider the set of linear bounded operators $\mathscr{L}$ from $\mathscr{H}$ to (or into) $\mathscr{H}$ and define the restricted matrix representation operator $\mathbf{M}_n(\widehat{L})$ whose argument is taken from $\mathscr{L}$. It creates $n \times n$ matrix representation on $\mathscr{H}_n$ for $\widehat{L}$, whose general term at the intersection of the ith row and jth column is $(u_i, \widehat{L}u_j)$. As can be easily proven, this operator has the following properties beyond its linearity.

$$\mathbf{M}_n\left(\widehat{P}^{(n)}\widehat{L}\right) = \mathbf{M}_n\left(\widehat{L}\widehat{P}^{(n)}\right) = \mathbf{M}_n\left(\widehat{L}\right), \quad \widehat{L} \in \mathscr{L} \tag{11}$$

$$\mathbf{M}_n\left(\widehat{L}_1\widehat{P}^{(n)}\widehat{L}_2\right) = \mathbf{M}_n\left(\widehat{L}_1\right)\mathbf{M}_n\left(\widehat{L}_2\right), \quad \widehat{L}_1, \widehat{L}_2 \in \mathscr{L} \tag{12}$$

$$\mathbf{M}_n\left(\left[\widehat{I}-\widehat{P}^{(n)}\right]\widehat{L}\right) = \mathbf{M}_n\left(\widehat{L}\left[\widehat{I}-\widehat{P}^{(n)}\right]\right) = \mathbf{0}, \quad \widehat{L} \in \mathscr{L} \tag{13}$$

where $\mathbf{0}$ stands for $n \times n$ zero matrix.

If we take $\widehat{L}$ as $\widehat{I}$, the unit operator of $\mathscr{H}$, in (11) and remember that the matrix representation of $\widehat{I}$ on $\mathscr{H}_n$ is $\mathbf{I}_n$, the $n \times n$ unit matrix, then we can write

$$\mathbf{M}_n\left(\widehat{P}^{(n)}\right) = \mathbf{I}_n \tag{14}$$

On the other hand, by setting $\widehat{L}_1 = \widehat{x}$ and $\widehat{L}_2 = \widehat{x}^k$ in (12) for natural number values of k, we can write

$$\mathbf{M}_n\left(\widehat{x}\,\widehat{P}^{(n)}\widehat{x}^k\right) = \mathbf{M}_n\left(\widehat{x}\right)\mathbf{M}_n\left(\widehat{x}^k\right), \quad k = 0, 1, 2, \ldots \tag{15}$$

which urges us to seek an approximate formula for the matrix representation of $\widehat{x}^k$. To this end we can attempt to construct a recursion at least approximately and write the following identity

$$\widehat{x}^{k+1} \equiv \widehat{x}\,\widehat{P}^{(n)}\widehat{x}^k + \widehat{x}\left[\,\widehat{I} - \widehat{P}^{(n)}\,\right]\widehat{x}^k, \qquad k = 0, 1, 2, \ldots \tag{16}$$

which implies the following matrix equality from (16)

$$\mathbf{M}\left(\widehat{x}^{k+1}\right) \equiv \mathbf{M}\left(\widehat{x}\right)\mathbf{M}\left(\widehat{x}^k\right) + \mathbf{M}\left(\widehat{x}\left[\,\widehat{I} - \widehat{P}^{(n)}\,\right]\widehat{x}^k\right), \quad k = 0, 1, 2, \ldots \tag{17}$$

The rightmost term of this equality contains the operator $[\widehat{I} - \widehat{P}^{(n)}]\widehat{x}^k$ whose norm decreases as n grows. For a specified precision, it is always possible to find a sufficiently high value of n, for which the second term of the right hand side becomes ignorable beside the first one. This implies that

$$\mathbf{M}_n^{(pr)}\left(\widehat{x}^{k+1}\right) \approx \mathbf{M}_n^{(pr)}\left(\widehat{x}\right)\mathbf{M}_n^{(pr)}\left(\widehat{x}^k\right), \quad k = 0, 1, 2, \ldots \quad n \geq n_{cr} \tag{18}$$

where the superscript (pr) specifies the utilized precision in the calculations and the subscript cr specifies the critical dimension, for n values greater than or equals to which, approximation becomes an equality within the precision (pr).

Equation 18 can be iteratively solved to give

$$\mathbf{M}_n^{(pr)}\left(\widehat{x}^k\right) \approx \mathbf{X}_n^{(pr)^k}, \quad k = 0, 1, 2, \ldots \quad n \geq n_{cr} \tag{19}$$

$$\mathbf{X}_n^{(pr)} \equiv \mathbf{M}_n^{(pr)}\left(\widehat{x}\right), \quad n \geq n_{cr} \tag{20}$$

Equation 19 is the basic source of the recently proven fluctuationlessness theorem [6]. As a matter of fact, the operator $[\widehat{I} - \widehat{P}^{(n)}]$ together with the powers of the operator $\widehat{x}$ describes the fluctuations (outgoing and incoming transitions) between $\mathscr{H}_n$ and its complement to $\mathscr{H}$. Hence the one or more than one appearances in the formulae can be considered as fluctuations and the approximations by ignoring them may be

called "fluctuationlessness case". In this point of view (19) is the fluctuationlessness approximation of the natural number power of the independent variable related algebraic multiplication operator.

At this point we assume that the function $f(x)$ is analytic in a region containing the interval $[a, b]$ in the x complex plane. This means that it can be expanded to the following series.

$$f(x) = \sum_{i=0}^{\infty} f_i x^i \tag{21}$$

which means

$$\widehat{f} = \sum_{i=0}^{\infty} f_i \widehat{x}^i \tag{22}$$

and

$$\mathbf{M}_n^{(pr)}\left(\widehat{f}\right) = \sum_{i=0}^{\infty} f_i \mathbf{M}_n^{(pr)}\left(\widehat{x}\right)^i = f\left(\mathbf{X}_n^{(pr)}\right) \tag{23}$$

This is the mathematical expression of the recently proven fluctuationlessness theorem [6] which has been used in many integration related problems [1–5,7–9] successfully. (Here, the references are all from the author's group. This is not because of overemphasizing on this group's works. Within the knowledge of the author and recent comprehensive literature, all these kinds of approaches seem to be quite new and they will attract other researchers concerning with these kind issues.) It states that the matrix representation of a function on $\mathscr{H}_n$ is equal to the image of the matrix representation of the independent variable on $\mathscr{H}_n$ under the function for sufficiently high n values within a prescribed calculation precision. Our purpose here is to use this fact for the evaluation of the matrix representation on $\mathscr{H}_n$ for a prescribed accuracy.

Paper is organised as follows. The second section covers the basic idea for the evaluation. The third section involves the illustrative numerical implementations. The fourth section including the concluding remarks finalizes the paper.

2 Basic idea behind the evaluation

Consider the subspaces $\mathscr{H}_n$ and $\mathscr{H}_m$ of $\mathscr{H}$ with $m > n$. We can write

$$\mathbf{M}_n^{(pr)}\left(\widehat{f}\right) = \mathbf{E}_{m\to n}^T \mathbf{M}_m^{(pr)}\left(\widehat{f}\right)\mathbf{E}_{m\to n} \tag{24}$$

where $\mathbf{E}_{m\to n}$ chops out the columns of the m dimensional matrix representation if their indices are greater than n. In other words, it is derived from the m-dimensional unit matrix by excluding all columns except the first n leftmost ones. Equation 24 is

exact and can be approximated via fluctuationlessness approximation as follows for m values greater than a critical value corresponding to a given precision

$$\mathbf{M}_n^{(pr)}\left(\widehat{f}\right) = \mathbf{E}_{m\to n}^T f\left(\mathbf{X}_m^{(pr)}\right)\mathbf{E}_{m\to n} \tag{25}$$

Now as m grows up to infinity $f(\mathbf{X}_m^{(pr)})$ approximates $\mathbf{M}_m^{(pr)}(\widehat{f})$ more precisely. Therefore we can construct an algorithm which evaluates the right hand side of (25) for various increasing m values starting from $m = n$ until the evaluations for two consecutive m values match within the prescribed precision.

Although there are various methods to evaluate the matrix $f(\mathbf{X}_m^{(pr)})$ and subroutines are available in certain computer programming or scripting languages we recommend to use the spectral resolution. To this end we need to evaluate the eigenpairs ($\xi_i^{(pr)}$ and $\mathbf{x}_i^{(pr)}$ denote the ith eigenvalue and the corresponding eigenvectors respectively). There are a lot of methods to this end. However, the important thing here is the evaluations are needed to be done only once. Then they can be stored for future uses like it is done for the weight and nodal parameters of the quadratures. In other words, there is a strong universality of the evaluations of the eigenpairs since they do not depend on the function $f(x)$. Another important thing about the eigenvalues evaluation is the procedure's ill-posedness which is increasing with growing m values. This comes from the locations of the eigenvalues. As can be proven via upper and lower bounds of a Rayleigh quotient constructed over the matrix $\mathbf{X}_n^{(pr)}$ the eigenvalues must be located at the interior points of the interval $[a, b]$. As m grows the number of the eigenvalues in this interval increases to result in distance decreases between two consecutive eigenvalues. The closer the consecutive eigenvalue couples the higher ill-posedness. This means that the ascending m values require increasing calculation precision. This demands flexibility in the precision increase and can be provided by symbolic programming languages like Mathematica, MuPAD or by some scientific libraries for C or C-like computer languages. High precision requirement and the ill-posedness may increase the computational complexity of the eigenvalue determination. However, the full procedure's computational complexity is not affected from this, since the eigenvalue determination is in fact outside the procedure and needed to be done just once.

Now by assuming the eigenpairs are evaluated and available for the procedure we can write

$$f\left(\mathbf{X}_m^{(pr)}\right) = \sum_{i=1}^{m} f\left(\xi_i^{(pr)}\right)\mathbf{x}_i^{(pr)}{\mathbf{x}_i^{(pr)}}^T \tag{26}$$

The utilization of this formula in (25) gives

$$\mathbf{M}_n^{(pr)}\left(\widehat{f}\right) = \sum_{i=1}^{m} f\left(\xi_i^{(pr)}\right)\mathbf{y}_i^{(pr)}{\mathbf{y}_i^{(pr)}}^T \tag{27}$$

where

$$\mathbf{y}_i^{(pr)} \equiv \mathbf{E}_{m\to n}^T \mathbf{x}_i^{(pr)}, \quad i = 1, \ldots, m \tag{28}$$

This completes the construction of the algorithm.

3 Illustrative implementations

In this section we give some numerical implementations for certain illustrative cases. In implementations we have used MuPAD Computer Algebra System and have chosen the interval ends as 0 and 1 without any loss of generality because an appropriate affine transformation can always bring us to the case of this very specific interval from any finite and closed interval. Our first example focuses on the function

$$f(x) \equiv \mathrm{e}^{\alpha x} \tag{29}$$

where α is used somehow to represent the curvature of the function. This function has no singularity at any finite region of the x complex plane. It has an essential singularity at infinity.

Since its analiticity extends towards infinity we can anticipate good quality in the approximation via our method here. This, in fact, happens. However, approximation quality decreases parallel to the increase in α as can be seen from Table 1.

The results in Table 1 are obtained within 50 digit accuracy and show that the convergence is very rapid in the case where $\alpha = 1$. For m values between 5 and 20 fifty correct digits are obtained. That is, each increment in m by 1 brings three more correct digits. However, this rapidity decreases as α climbs to 20. The average correct digit gain for an increment by 1 in m decreases to almost 1 in the last column. This is an expected negative behavior. However, by appropriately increasing m and the precision in the calculations appropriately one can always get any desired accuracy.

Our second example focuses on the case where $f(x) \equiv \cos \alpha x$. This function has also no singularity in any finite region of x complex plane. However it is not monotonously increasing function on the interval $[a, b]$ for all α values. Its oscillatory behavior increases as α grows. The results for this case are given in Table 2.

First three columns show almost same behavior of the previous case. However, the last column values may be considered as the signal of the negative effect of high oscillations on the approximation quality. However it is not so clear as much as the

Table 1 Relative error with respect to the exact value in the Frobenius norm of the approximated five dimensional ($n = 5$) matrix representations with a few m values for several curvature parameter values in the case of $f(x) = \mathrm{e}^{\alpha x}$

m	$\alpha = 1$	$\alpha = 5$	$\alpha = 10$	$\alpha = 20$
5	1.298×10^{-2}	1.347×10^{-1}	2.408×10^{-1}	4.015×10^{-1}
10	5.166×10^{-17}	3.227×10^{-9}	1.992×10^{-6}	3.024×10^{-4}
15	2.111×10^{-35}	1.217×10^{-20}	6.493×10^{-15}	5.515×10^{-10}
20	8.449×10^{-51}	4.755×10^{-34}	2.432×10^{-25}	1.646×10^{-17}

Table 2 Relative error with respect to the exact value in the Frobenius norm of the approximated five dimensional ($n = 5$) matrix representations with a few m values for several curvature parameter values in the case of $f(x) = \cos(\alpha x)$

m	$\alpha = 1$	$\alpha = 5$	$\alpha = 10$	$\alpha = 20$
5	1.445×10^{-2}	2.986×10^{-1}	5.090×10^{-1}	1.122×10^{-1}
10	5.978×10^{-17}	1.599×10^{-8}	2.146×10^{-5}	5.699×10^{-2}
15	2.453×10^{-35}	6.719×10^{-20}	8.908×10^{-14}	5.732×10^{-7}
20	1.593×10^{-51}	2.738×10^{-33}	3.729×10^{-24}	3.356×10^{-14}

effect of the curvature on the quality. To clarify this point we have investigated the case where $\alpha = 50$, $n = 5$, $m = 20$ for the functions $e^{\alpha x}$ and $\cos \alpha x$. The former function have given 2.056×10^{-9} in the Frobenius norm relative error while the latter one is giving the value 1.102×10^{-2}. This apparently shows the negative influence of increasing oscillations in the value of the function on the approximation quality. However, the curvature dependence seems to be more sensitive than the oscillation dependence for the approximation quality.

Our third example is directed to understand the effect of the polar singularity location on the approximation quality. We choose

$$f(x) \equiv \frac{1}{1 + \frac{\alpha}{\beta} x} \tag{30}$$

which has just a single pole located at $x = -\beta/\alpha$ as the only singularity. Here we have used two parameters deliberately to somehow distinguish the curvature and singularity location dependences. When α, the curvature parameter, is 1, $-\beta$ is the location of the singularity. Hence we call it the parameter for the singularity location on the negative real axis or briefly "singularity parameter". On the other hand, the case where $\beta = 1$ leaves α alone to describe the curvature. Hence we call α curvature parameter. That is, $\alpha = 1$ and $\beta = 1$ normalizations are used to define the singularity location parameter and curvature parameters respectively. Except the point where $x = -\beta/\alpha$ the function in (30) is analytic everywhere in the x complex plane. For a fixed positive α value the pole is at minus infinity when β is at infinity and it moves from there to $-1/\alpha$ as β diminishes down to 1. Therefore it gets closer to 0 which is the left end of the interval under consideration. This urges us to anticipate increasing negative effect on the approximation quality when β approaches from right to 1. Table 3 is about the investigation of the increasing curvature's effect on the approximation quality. There is nothing unexpected except the shift in α to the left to get the similar quality behavior of the two previous cases.

As can be noticed immediately, the quality is more sensitive to curvature than before. Table 4 is constructed for the case where $\alpha = 20$, $n = 5$, $m = 20$ for several β values to reveal the positive effect of the increasing β values on the quality.

Our fourth example focuses on the simplest case of the branch point singularity. We concern with the following function which has two branch points, one at infinity and one at a finite point of the independent variable's complex plane.

Table 3 Relative error with respect to the exact value in the Frobenius norm of the approximated five dimensional ($n = 5$) matrix representations for several curvature parameter values in the case of $f(x) = \frac{1}{1+\frac{\alpha}{\beta}x}$ when $\beta = 1$

m	$\alpha = 0.1$	$\alpha = 1$	$\alpha = 10$	$\alpha = 20$
5	2.564×10^{-4}	1.338×10^{-2}	1.508×10^{-1}	2.444×10^{-1}
10	1.543×10^{-20}	3.027×10^{-10}	3.121×10^{-4}	3.079×10^{-3}
15	9.154×10^{-37}	6.747×10^{-18}	6.274×10^{-7}	3.720×10^{-5}
20	5.411×10^{-53}	1.498×10^{-25}	1.253×10^{-9}	4.449×10^{-7}

Table 4 Relative error with respect to the exact value in the Frobenius norm of the approximated five dimensional ($n = 5$) matrix representations for several β values in the case of $f(x) = \frac{1}{1+\frac{\alpha}{\beta}x}$ when $\alpha = 20$

m	$\beta = 1$	$\beta = 10$	$\beta = 20$	$\beta = 50$
5	2.444×10^{-1}	3.304×10^{-2}	1.338×10^{-2}	3.186×10^{-3}
10	3.079×10^{-3}	6.464×10^{-8}	3.027×10^{-10}	5.642×10^{-14}
15	3.720×10^{-5}	1.244×10^{-13}	6.747×10^{-18}	9.851×10^{-25}
20	4.449×10^{-7}	2.384×10^{-19}	1.498×10^{-25}	1.713×10^{-35}

Table 5 Relative error with respect to the exact value in the Frobenius norm of the true and approximated five dimensional ($n = 5$) matrix representations for several curvature parameter values in the case of $f(x) = \sqrt{1 + \frac{\alpha}{\beta}x}$ when $\beta = 1$

m	$\alpha = 0.1$	$\alpha = 1$	$\alpha = 10$	$\alpha = 20$
5	3.204×10^{-5}	1.640×10^{-3}	1.452×10^{-2}	1.996×10^{-2}
10	1.081×10^{-22}	2.103×10^{-12}	1.917×10^{-6}	1.719×10^{-5}
15	2.546×10^{-39}	1.864×10^{-20}	6.274×10^{-7}	8.650×10^{-8}
20	8.725×10^{-56}	2.348×10^{-28}	1.578×10^{-9}	5.993×10^{-10}

$$f(x) \equiv \sqrt{1 + \frac{\alpha}{\beta}x} \tag{31}$$

where α and β stand for the curvature parameter and the singularity location parameter respectively as we defined before (Table 5).

The results in this table do not show anything peculiar to this case. They behave quite similar to the ones in the case of simple polar singularity. Of course there will be certain differences in the obtained accuracy for a given value of α from function to function.

Same thing is true for also for the convergences corresponding to different β values for a fixed curvature parameter value (α). These are given in Table 6.

Table 6 Relative error with respect to the exact value in the Frobenius norm of the approximated five dimensional ($n = 5$) matrix representations for several β values in the case of $f(x) = \sqrt{1 + \frac{\alpha}{\beta}x}$ when $\alpha = 20$

m	$\beta = 1$	$\beta = 10$	$\beta = 20$	$\beta = 50$
5	1.996×10^{-2}	3.928×10^{-3}	1.640×10^{-3}	3.964×10^{-4}
10	1.719×10^{-5}	4.428×10^{-10}	2.103×10^{-12}	3.946×10^{-16}
15	8.650×10^{-8}	3.402×10^{-16}	1.864×10^{-20}	2.736×10^{-27}
20	5.993×10^{-10}	3.706×10^{-22}	2.348×10^{-28}	2.698×10^{-38}

Our next examples is chosen to test the case of branch point singularity with infinite Riemann sheets.

$$f(x) \equiv \ln\left(1 + \frac{\alpha}{\beta}x\right) \tag{32}$$

where α and β can be considered as the curvature and singularity location parameters respectively as we did before. The results for this case are given in Tables 7 and 8.

Table 7 does not imply any noticeable effect of the Riemann sheets' number on the convergence rate. Of course, there will be slight differences in the convergence from function to function and there are signals of this in the comparison of Tables 5 and 7.

Until now, we have dealt with the case where the finite branch point is located to the left of the interval. We could deal with the cases where the finite branch point is

Table 7 Relative error with respect to the exact value in the Frobenius norm of the approximated five dimensional ($n = 5$) matrix representations for several curvature parameter values in the case of $f(x) = \ln\left(1 + \frac{\alpha}{\beta}x\right)$ when $\beta = 1$

m	$\alpha = 0.1$	$\alpha = 1$	$\alpha = 10$	$\alpha = 20$
5	2.220×10^{-3}	1.529×10^{-2}	4.488×10^{-2}	5.366×10^{-2}
10	2.227×10^{-20}	5.838×10^{-11}	1.803×10^{-5}	1.424×10^{-4}
15	7.208×10^{-37}	7.111×10^{-19}	2.034×10^{-8}	9.797×10^{-7}
20	2.929×10^{-53}	1.086×10^{-26}	2.825×10^{-11}	8.210×10^{-9}

Table 8 Relative error with respect to the exact value in the Frobenius norm of the approximated five dimensional ($n = 5$) matrix representations for several β values in the case of $f(x) = \ln\left(1 + \frac{\alpha}{\beta}x\right)$ when $\alpha = 20$

m	$\beta = 1$	$\beta = 10$	$\beta = 20$	$\beta = 50$
5	5.366×10^{-2}	2.331×10^{-2}	1.529×10^{-2}	7.671×10^{-3}
10	1.424×10^{-4}	7.849×10^{-9}	5.838×10^{-11}	2.271×10^{-14}
15	9.79×10^{-7}	3.284×10^{-15}	7.111×10^{-19}	2.167×10^{-25}
20	8.210×10^{-9}	1.094×10^{-20}	1.086×10^{-26}	2.588×10^{-36}

located to the right of the interval. In fact, we did and noticed nothing different in the convergence behavior except slight changes from function to function. We find it to report those results here unnecessary.

Table 8 shows that there is a slight worsening in the convergence for this case. That is, one can conjecture that there is a slight negative effect of the infinity in the number of the Riemann sheets. In other words, the convergence is slightly sensitive to the number of Riemann sheets beside the dependence on the curvature or the singularity location.

This example finalizes this section, although the number of the examples could be increased to cover some other secondary or maybe much less important cases.

4 Concluding remarks

Author believes that what we have stated in the previous sections covers all primary remarks and the rest are secondary items which may be skipped. The method is simple and universal and it is possible to get any desired accuracy in the matrix representation approximation just by appropriately choosing two parameters, m and pr which are the internal matrix dimension and the precision in the calculations respectively. The approximation power of the method comes from the fluctuationlessness approximation and its convergence rate is remarkably good in many moderate cases. What we have obtained here were limited to univariate cases. However, the structure of the method allows us to extend it to the multivariate cases by paying sufficient care to the construction of multivariate basis functions.

Another important thing is numerical integration. If we choose $n = 1$ and the weight function is taken normalized to have unit integral then the only matrix representation element becomes the integral of $f(x)$. In this case m different x values will form the set of nodes where the function values are evaluated. The corresponding weight parameter to each node will be the square of the first element of the eigenvector corresponding to the eigenvalue which is that node in fact. This gives an m-node quadrature like formula. The comparison of the result of this formula with the standing quadratures and methods like Gauss quadratures and Romberg method for same number of nodes revealed the fact that this method may become apparently superior to these methods when appropriate basis set is used although we have not given any result since it is outside the scope of this paper. This fact is due to very possibly the natural selections of the nodes to suppress the error coming from fluctuations. One can develop methods or formulae to estimate the error and then compare with the ones of the existing approaches, to prove the superiority. This may be a future concern for interested readers involving us.

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